International Journal of Theoretical Physics, Vol. 16, No. 7 (1977), pp. 541–549

A Characterization of Carter's Separable Space-Times

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Received: 11 *April* 1977

Abstract

In a space-time admitting a two-parameter Abelian isometry group, and a quadratic Killing tensor with the eigenvalues $(\lambda, \lambda, \mu, \mu)$ and vanishing Lie derivatives with respect to the Killing vectors, we construct a canonical coordinate system. The isometry group acts orthogonally transitively. The Hamilton–Jacobi equation is separable. We give a necessary and sufficient condition for the separability of the Klein–Gordon equation. We obtain Carter's space-times with completely separable Klein–Gordon equation.

1. *Introduction*

Carter (1968) has discussed space-times admitting a two-parameter Abelian isometry group acting orthogonally transitively (Carter, 1969), and fulfilling the additional condition that the Hamilton-Jacobi equation is soluble by separation of variables (Woodhouse, 1975). The separability gives rise to a separation constant that proves to be a quadratic constant of motion generated by a quadratic Killing tensor (Penrose and Walker, 1970). This Killing tensor has (a) the eigenvalues $(\lambda, \lambda, \mu, \mu)$, (b) vanishing Lie derivatives with respect to the two commuting Killing vectors, and (c) is not reducible[1] to the commuting Killing vectors and the metric.

Carter's stronger condition that the Schrödinger equation also separates is equivalent to the condition that the Ricci tensor component $R_{ab} V^a W^b$ vanishes, where V^a and W^a are directions parallel to the two ignorable coordinate vectors used by Carter (see Dietz, 1976, Theorem 3).

In this paper, we solve the "inverse Carter problem." We consider the Killing vectors and tensors of the space-time as the fundamental objects. We assume the

[1] We distinguish between "reducibility" and "reducibility to a given set of Killing vectors": (a) A Killing tensor is reducible iff it can be written as a linear combination of symmetric products of Killing vectors and of the metric with constant coefficients (Penrose and Walker, 1970). (b) A Killing tensor is reducible to the r Killing vectors ξ_j^a ($j = 1, \ldots, r$) iff it can be written as a linear combination of symmetric products of these Killing vectors ξ_j^a and the metric with constant coefficients.

existence of a two-parameter Abelian isometry group and a Killing tensor of order 2 with the properties (a)-(c). In Section 2 we give the Killing equations in tetrad form for later use. In Section 3 we show that the conditions (a)-(c) make only one geometric situation possible, which we describe in terms of intersections between the eigenspaces of the Killing tensor and the group orbits. We construct canonical coordinate vectors and show that the assumed isometry group acts orthogonally transitively. In Section 4 we show the separability of the Hamilton-Jacobi, and the Klein-Gordon equation if additionally $R_{ab}V^aW^b = 0$. The metric components evaluated in the selected coordinates are identical with those derived by Carter (1968). Our calculation holds for arbitrary[2] λ and μ.

2. *Killing Vectors and Killing Tensors*

A vector field ξ^a satisfying the Killing equation

$$[\xi, g]^{ab} = 0 \tag{2.1}$$

defines a Killing vector field, where [,] denotes the Nijenhuis bracket operation in local coordinates defined by (Geroch, 1970)

$$[A, B]^{a\cdots h} := pA^{k(a\cdots c}\nabla_k B^{d\cdots h)} - qB^{k(a\cdots d}\nabla_k A^{e\cdots h)}$$

where A and B are symmetric contravariant tensors of order p and q, respectively, and ∇ is any torsion-free connection.

We generalize the Killing equation: A symmetric second-order tensor is called a quadratic Killing tensor if

$$[K, g]^{abc} = 0 \tag{2.2}$$

(M, g) is a space-time with signature $(+ - - -)$ admitting a two-parameter isometry group generated by the Killing vectors $\xi_i{}^a$ with non-null group orbits, and a quadratic Killing tensor K^{ab} satisfying

$$[\xi_1, \xi_2]^a = 0 \tag{2.3}$$

$$[K, \xi_i]^{ab} = 0 \tag{2.4}$$

K^{ab} is not reducible to $\xi_i{}^a$ and g^{ab} and possesses two twofold degenerated eigenvalues λ and μ.[3] We can choose an orthonormal frame

$$\underset{\alpha}{E}^a = (T^a, X^a, Y^a, Z^a)$$

[2] If λ and μ are independent functions, they may choose to be coordinates (cf. Corollary 2). If additionally the condition (4.1) imposed on the Ricci tensor holds equivalent with $R^{ab}\partial_a\lambda\partial_b\mu = 0$, Hauser and Malhiot (1976) have shown that then the existence of the Killing tensor K^{ab} implies the existence of the two-parameter Abelian isometry group generated by ξ_i^a, which acts orthogonally transitively and fulfills $[\xi_i, K]^{ab} = 0$.

[3] We exclude the degenerated case $\lambda = \mu$ because then in consequence of $[K, g]^{abc} = 0$

$$K^{ab} = \text{const}\cdot g^{ab}$$

such that K^{ab} can be written as

$$K^{ab} = \lambda(T^a T^b - X^a X^b) - \mu(Y^a Y^b + Z^a Z^b) \tag{2.5}$$

where λ and μ are the eigenvalues of K^{ab}. Let the eigenspace of λ be timelike. One frame vector has to be timelike, say T^a. The metric tensor is

$$g^{ab} = T^a T^b - X^a X^b - Y^a Y^b - Z^a Z^b \tag{2.6}$$

For the given Killing tensor (2.5) the eigenvectors (T^a, X^a, Y^a, Z^a) are not unique since we have the freedom of a boost in the TX eigenspace and a spacelike rotation in the YZ eigenspace (Eisenhart, 1948, p. 110). We obtain, equivalently to Eisenhart (1948, p. 129),

$$T(\lambda) = X(\lambda) = 0 = Y(\mu) = Z(\mu) \tag{2.7a}$$

$$\left.\begin{aligned} T(\ln|\lambda - \mu|) &= 2[T, Z]^a Z_a \\ X(\ln|\lambda - \mu|) &= 2[X, Y]^a Y_a \\ Y(\ln|\lambda - \mu|) &= 2[Y, X]^a X_a \\ Z(\ln|\lambda - \mu|) &= 2[T, Z]^a T_a \end{aligned}\right. \tag{2.7b}$$

$$\left.\begin{aligned} [Y, X]^a X_a + [Y, T]^a T_a &= 0 = [Z, X]^a X_a + [Z, T]^a T_a \\ [T, Y]^a Y_a - [T, Z]^a Z_a &= 0 = [X, Y]^a Y_a - [X, Z]^a Z_a \\ [Y, T]^a X_a + [Y, X]^a T_a &= 0 = [Z, T]^a X_a + [Z, X]^a T_a \\ [T, Z]^a Y_a + [T, Y]^a Z_a &= 0 = [X, Z]^a Y_a + [X, Y]^a Z_a \end{aligned}\right. \tag{2.7c}$$

for the Killing tensor equation (2.2) using (2.5) and (2.6). For a Killing vector ξ^a we get from (2.1)

$$[\xi, \underset{\alpha}{E}]^a \underset{\beta}{E}_a + [\xi, \underset{\beta}{E}]^a \underset{\alpha}{E}_a = 0 \tag{2.8}$$

3. *A Canonical Coordinate System*

The space-time considered admits a two-parameter Abelian isometry group generated by $\xi_i{}^a$. The group orbits are two-dimensional surfaces. The Killing tensor (2.5) defines two two-dimensional eigenspaces spanned by T^a, X^a and Y^a, Z^a, which are orthogonal. The eigenspaces define two-surface elements locally. If we investigate all possibilities for intersections between the Killing tensor eigenspaces and the group orbits, which can be time- or spacelike, we find that only two cases can be realized because of reasons connected with the dimension and the timelike, spacelike, or null character of the surface elements, orbits, and their intersections:

A. The orbits are timelike.
   1. The orbits and the TX eigenspaces of K^{ab} coincide locally.
   2. The orbits intersect the TX eigenspaces in a one-dimensional timelike and the YZ eigenspaces in a one-dimensional spacelike intersection.

B. The orbits are spacelike.
   1. The orbits and the YZ eigenspaces coincide locally.
   2. The orbits intersect the TX and the YZ eigenspaces in a one-dimensional spacelike intersection.

> *Lemma.* In the given space-time, the Killing tensor K^{ab} is reducible to $\xi_i{}^a$ and g^{ab}, if one eigenspace of K^{ab} coincides locally with the orbits.

Proof. The proofs for A1 and B1 are completely analogous. We consider case A1: We use the freedom of a boost in the TX eigenspace of K^{ab} (see page 543) to turn T^a in the direction of the (say timelike) Killing tensor $\xi_1{}^a$:

$$\xi_1{}^a = \tau T^a \quad \text{and} \quad \xi_2{}^a = \alpha T^a + \beta X^a \tag{3.1}$$

After evaluating the equations (2.8) for $\xi_i{}^a$ together with (2.7) we have $\mu = \text{const}$,

$$Y(\ln|\tau|) = Y(\ln|\alpha|) = Y(\ln|\beta|) = Y(\ln|\lambda - \mu|^{1/2})$$

and vanishing derivations in the directions of T^a, X^a, and Z^a. We integrate and obtain, with constants c_α, c_β, and c_λ,

$$\tau = c_\alpha \alpha = c_\beta \beta = c_\lambda |\lambda - \mu|^{1/2}$$

from which it follows that

$$K^{ab} = c_1 \xi_1{}^a \xi_1{}^b + c_2 \xi_2{}^a \xi_2{}^b + c_3 \xi_1{}^{(a} \xi_2{}^{b)} + \mu g^{ab}$$

where c_1, c_2, and c_3 are combinations of c_α, c_β, and c_λ. (We have to remark: We get the same expression for K^{ab} if $\xi_1{}^a$ is spacelike.) Because of our assumption that the given Killing tensor K^{ab} is not reducible to $\xi_i{}^a$ and g^{ab} we exclude A1 and B1. A2 and B2 remain to be investigated.

> *Theorem 1.* If in a space-time (M, g) admitting (a) a two-parameter Abelian isometry group generated by $\xi_i{}^a$ $(i = 1, 2)$, (b) a quadratic Killing tensor K^{ab} with the eigenvalues $(\lambda, \lambda, \mu, \mu)$ commuting with $\xi_i{}^a$, the eigenspaces do not coincide locally with the group orbits, then the vectors $\xi_1{}^a, MX^a, NY^a, \xi_2{}^a$ commute in pairs, where $M^2 = (\lambda - \mu) \cdot \Delta_x^{-1}$ and $N^2 = (\lambda - \mu)\Delta_y^{-1}$, with functions Δ_x and Δ_y fulfilling $T(\Delta_x) = Z(\Delta_x) = Y(\Delta_x) = 0$; $T(\Delta_y) = Z(\Delta_y) = X(\Delta_y) = 0$ so that M^2 and N^2 are positive.

Proof. Without loss of generality we give the proof for the timelike orbits (A2).[4] We use the freedom of a boost in the TX eigenspaces to turn T^a in the direction of the timelike intersections between the orbits and the TX eigenvectors, the freedom of a spacelike rotation to turn Z^a in the direction of the spacelike intersections between the orbits and the spacelike eigenspaces of K^{ab} such that

$$\xi_i{}^a = \alpha_i T^a + \beta_i Z^a \tag{3.2}$$

[4] In the case of spacelike orbits we substitute T by X and vice versa.

with function α_i, β_i satisfying $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$. We evaluate equations (2.8) and (2.4) for $\xi_i{}^a$ and obtain

$$\xi_i(\lambda) = 0 = \xi_i(\mu) \tag{3.3a}$$

because

$$[\xi_i, \underset{\alpha}{E}]^a \underset{\alpha}{E}_a = 0$$

[cf. equation (2.8)],

$$[\xi_i, T]^a X_a + [\xi_i, X]^a T_a = 0 = [\xi_i, Y]^a Z_a + [\xi_i, Z]^a Y_a \tag{3.3b}$$

and

$$[\xi_i, T]^a Y_a + [\xi_i, Y]^a T_a = 0 = \lambda[\xi_i, T]^a Y_a + \mu[\xi_i, Y]^a T_a$$

$$[\xi_i, T]^a Z_a + [\xi_i, Z]^a T_a = 0 = \lambda[\xi_i, T]^a Z_a + \mu[\xi_i, Z]^a T_a$$

$$[\xi_i, X]^a Y_a + [\xi_i, Y]^a X_a = 0 = \lambda[\xi_i, X]^a Y_a + \mu[\xi_i, Y]^a X_a$$

$$[\xi_i, X]^a Z_a + [\xi_i, Z]^a X_a = 0 = \lambda[\xi_i, X]^a Z_a + \mu[\xi_i, Z]^a X_a$$

Since $\lambda - \mu \neq 0$ (see footnote 3 above), all these commutator components vanish. Therefore and with

$$[\xi_i, \underset{\alpha}{E}]^a \underset{\alpha}{E}_a = 0$$

[see (2.8)] we find

$$[\xi_i, T]^a \sim X^a, \qquad [\xi_i, X]^a \sim T^a, \qquad [\xi_i, Y]^a \sim Z^a, \qquad [\xi_i, Z]^a \sim Y^a$$

where we insert the tetrad form (3.2) of ξ_i and transvect the first relation with Y_a and the last relation with X_a in order to obtain

$$[T, Z]^a X_a = 0 = [T, Z]^a Y_a$$

That implies $[\xi_i, T]^a X_a = 0 = [\xi_i, Z]^a Y_a$ and with (3.3b)

$$[\xi_i, \underset{\alpha}{E}]^a = 0 \tag{3.4}$$

We express the tetrad vectors T^a and Z^a because of (3.2) as linear combinations of the Killing vectors $\xi_i{}^a$ and find from (3.4)

$$[Y, T]^a X_a = 0 = [Y, Z]^a X_a$$

$$[X, T]^a Y_a = 0 = [X, Z]^a Y_a$$

and because of the Killing tensor equation (2.7c)

$$[X, Y]^a T_a = 0 = [X, Y]^a Z_a$$

so that

$$[X, Y]^a = EX^a + FY^a \tag{3.5}$$

with $E = [X, Y]^a X_a$ and $F = [X, Y]^a Y_a$.

Since T^a and Z^a depend linearly on the Killing vectors $\xi_i{}^a$, we try to take two vectors parallel to X^a and Y^a together with $\xi_i{}^a$ as coordinate vectors. From the Killing tensor equation (2.7b) we find

$$2E = Y(\ln|\lambda - \mu|) \quad \text{and} \quad 2F = -X(\ln|\lambda - \mu|)$$

so that

$$[MX, NY]^a = 0 \Leftrightarrow \begin{cases} X(N^2(\lambda - \mu)^{-1}) = 0 \\ Y(M^2(\lambda - \mu)^{-1}) = 0 \end{cases} \tag{3.6}$$

This is fulfilled if

$$M^2 = (\lambda - \mu)\Delta_x^{-1} \quad \text{and} \quad N^2 = (\lambda - \mu)\Delta_y^{-1} \tag{3.7}$$

where Δ_x and Δ_y are functions with the assumed properties.

Additionally we obtain from (3.3)

$$\xi_i(M) = 0 = \xi_i(N) \tag{3.8}$$

and because of (3.4)

$$[\xi_i, MX]^a = 0 = [\xi_i, NY]^a$$

Therefore the vectors $(\xi_i{}^a, MX^a, NY^a)$ commute in pairs and can be chosen as a coordinate basis in (M, g). Let (t, x, y, ϕ) be the associated coordinates where t and ϕ are adapted to the Killing vectors $\xi_i{}^a$ and x, y are orthogonal coordinates unique up to transformations of themselves.

Corollary 1. The isometry group acts orthogonally transitively.

Proof. X^a and Y^a span two-dimensional surfaces in consequence of (3.5) which are orthogonal to the orbits spanned by $\xi_i{}^a$ because of (3.2).

Corollary 2. If the eigenvalues λ and μ of the given Killing tensor K^{ab} are functionally independent, it is possible to choose λ and μ as coordinates.

This is clear from (2.7a) and (2.7b), because $T(\mu) = 0 = Z(\lambda)$ follows from $[T, Z]^a T_a = 0 = [T, Z]^a Z_a$.

Hauser and Malhiot (1976) have chosen these coordinates (see footnote 2 above).

4. *Separability and Equivalence with Carter's Metrics*

It is well known that the Hamilton-Jacobi and Klein-Gordon equations separate with respect to coordinates adapted to commuting Killing vectors such that these equations separate with respect to t and ϕ (see Woodhouse, 1975; and Dietz, 1976). The Hamilton-Jacobi equation is also separable with respect to the orthogonal coordinates x or y: The Killing tensors K^{ab}, $\xi_1{}^a \xi_1{}^b$,

$\xi_2{}^a\xi_2{}^b$, and g^{ab} commute in pairs, have the common eigenvector ∂_x or ∂_y, and the associated quadratic constants of motion are independent functions. These are sufficient conditions for x or y to be an orthogonal separable coordinate for the Hamilton-Jacobi equation (Woodhouse, 1975). Therefore the Hamilton-Jacobi equation separates completely with respect to the given coordinates (t, x, y, ϕ).

Theorem 8 of Dietz (1976) ensures also the separability of the Klein-Gordon equation with respect to x or y if additionally the Ricci-tensor component R_{xy} vanishes, or equivalently (see footnote 2 above)

$$R_{ab}X^aY^b = 0 \tag{4.1}$$

If we use an invariant characterization of the eigenvectors X^a and Y^a, we obtain the following:

Theorem 2. In a space-time (M, g) admitting a two-parameter Abelian isometry group generated by $\xi_i{}^a$, and a Killing tensor K^{ab} of second order with eigenvalues $(\lambda, \lambda, \mu, \mu)$ and vanishing Lie derivatives with respect to $\xi_i{}^a$ the Klein-Gordon equation separates completely if

$$R_{ab}V^aW^b = 0$$

where V^a and W^a are different eigenvectors of K^{ab} orthogonal to the group orbits.

Finally, we obtain the same explicit form of the metric components depending on the coordinates x and y as Carter (1968) for his Klein-Gordon separable metrics: Define U by

$$U := \lambda(y) - \mu(x) \tag{4.2}$$

Then the relation between the tetrad and coordinate vectors is given by

$$\begin{aligned} T^a &= A_1\delta_t{}^a + B_1\delta_\phi{}^a \qquad & Z^a &= A_2\delta_t{}^a + B_2\delta_\phi{}^a \\ X^a &= (\Delta_x U^{-1})^{1/2}\delta_x{}^a, \qquad & Y^a &= (\Delta_y U^{-1})^{1/2}\delta_y{}^a \end{aligned} \tag{4.3}$$

because of (3.7), (4.2), with

$$\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} := \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}^{-1}$$

and

$$\operatorname{sig} U = \operatorname{sig} \Delta_x = \operatorname{sig} \Delta_y \tag{4.4}$$

With (2.6) and (4.3) the metric takes the form

$$\begin{aligned} g^{ab} = {} & (A_1{}^2 - A_2{}^2)\delta_t{}^a\delta_t{}^b + 2(A_1B_1 - A_2B_2)\delta_t^{(a}\delta_\phi^{b)} + (B_1{}^2 - B_2{}^2)\delta_\phi{}^a\delta_\phi{}^b \\ & - (U^{-1}\Delta_x)\delta_x{}^a\delta_x{}^b - (U^{-1}\Delta_y)\delta_y{}^a\delta_y{}^b \end{aligned} \tag{4.5}$$

where $Ug^{xx} = -\Delta_x(x)$ and $Ug^{yy} = -\Delta_y(y)$.

Because ∂_y is an eigenvector of the Killing tensor K^{ab} for the eigenvalue μ, we find [for a proof see Woodhouse (1975), Proposition 3.4]

$$\begin{aligned} 0 &= [\partial_y, K - \mu G]^{ab} \\ &= \partial_y(UA_1{}^2)\delta_t{}^a\delta_t{}^b + \partial_y(UB_1{}^2)\delta_\phi{}^a\delta_\phi{}^b + 2\partial_y(UA_1B_1)\delta_t^{(a}\delta_\phi^{b)} \end{aligned}$$

from which it follows that $UA_1{}^2$, $UB_1{}^2$, and UA_1B_1 are independent of y. Analogously $UA_2{}^2$, $UB_2{}^2$, and UA_2B_2 are independent of x. Therefore $V_x{}^r$ and $V_y{}^r$ $(r = 1, 2)$ defined by

$$V_x{}^r = |U|^{1/2}(A_1, B_1), \qquad V_y{}^r = |U|^{1/2}(A_2, B_2)$$

depend only on x and y, respectively, such that the metric (4.5) takes the form

$$\left(\frac{\partial}{\partial s}\right)^2 = U^{-1}\left[\overset{-}{(+)}\,\Delta_x\left(\frac{\partial}{\partial x}\right)^2 - \Delta_y\left(\frac{\partial}{\partial y}\right)^2 \overset{+}{(-)} \left(V_x{}^r\frac{\partial}{\partial z^r}\right)^2 - \left(V_y{}^r\frac{\partial}{\partial z^r}\right)^2\right]$$

where $r = 1, 2$ and $z^r = (t, \phi)$.[5] That is exactly the form of the Hamilton–Jacobi separable metrics given by Carter [1968, equation (57)].

The author (1976) has shown that condition (4.1) is equivalent with $\partial_x\partial_y \ln |g^{xx}|g|^{1/2}| = 0$, where $g = \det g_{ab}$. We integrate and obtain with arbitrary functions $f_x(x)$ and $f_y(y)$

$$|g|^{1/2} = Uf_xf_y$$

We use the freedom of transformation of x and y to get

$$|g|^{1/2} = U \tag{4.6}$$

which is equivalent to (4.1). The metric determinant g is given by

$$g = -U^2[\Delta_x\Delta_y(A_1B_2 - A_2B_1)^2]^{-1}$$

We compare the last two equations and find

$$U^2 = U^2(A_1B_2 - A_2B_1)^2\Delta_x\Delta_y \tag{4.7}$$

At least we define $Z_x{}^r$ and $Z_y{}^r$ by

$$Z_x{}^r := |\Delta_x|^{1/2}|U|^{1/2}(B_1, A_1), \qquad Z_y{}^r := |\Delta_y|^{1/2}|U|^{1/2}(B_2, A_2)$$

so that with (4.4) and (4.7) condition (4.6) is equivalent to

$$Z := \det Z_i^r = U$$

which is Carter's condition (79), so that the Klein–Gordon equation also separates completely.

Acknowledgments

I am grateful to Dr. R. Ebert and Dr. R. Rüdiger for valuable discussions and helpful comments.

[5] The signs in brackets are valid for spacelike orbits.

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